Role of conditional entropy in experiments of Landauer principle

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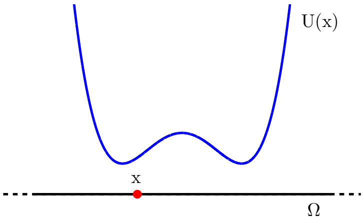
Outline

Bit Coding

General Framework for 1-Dim Systems



- 1-Dimensional system
- Each x value is a microstate
- Ω is the set of all possible microstates
- $U(x)$ bistable and symmetric potential

Bit Coding

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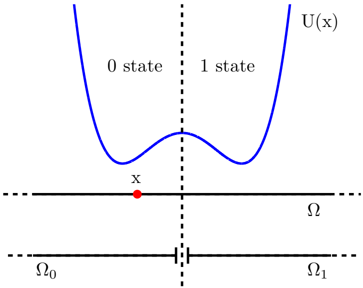

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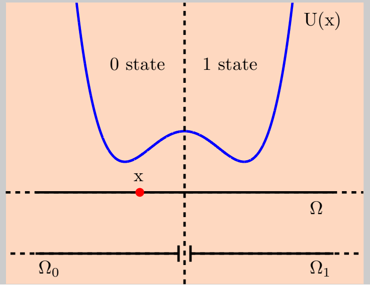
- Subdivide Ω in Ω_0 and Ω_1
- $x \in \Omega_0, \Omega_1 \equiv$ bit in the $0, 1$ state



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- Heath bath a temperature T

x fluctuates $\Rightarrow$ We describe its stochastic properties through a Probability Density Function $P(x)$.
We than define the probabilities of being in the 0-1 logic states

$$P_0 = \int_{\Omega_0} P(x)dx, \quad P_1 = \int_{\Omega_1} P(x)dx$$

Entropies and Landauer Principle

Gibbs thermodynamical entropy

$$S_G = -K_B \int_\Omega P(x) \log P(x) dx$$

Shannon information entropy

$$S_S = -K_B \sum_{i=0,1} P_i \log P_i$$

Landauer Principle

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$$Q \geq -T\Delta S_G \qquad (1)$$
$$\Delta S_G = \Delta S_S + \Delta S_{cond}$$

Warning1: Up to now $\Delta S_{cond} = 0$ was used with no clear justification.
Warning2: There is no simple representation of S_{cond} and ΔS_{cond}.

(1) T. Sagawa *J. Stat. Mech.* (2014) P03025

The sample structure of $P(x)$

For a system with bistable $U(x)$, it is reasonable that, in experiments, non-equilibrium $P(x)$ are functions with two peaks. We write

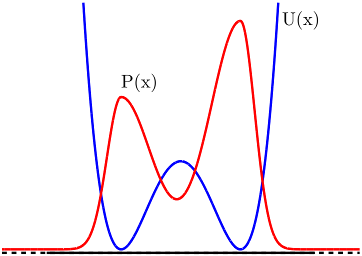
$$P(x) = P_a \eta_a(x) + P_b \eta_b(x)$$

η_a, η_b properties:

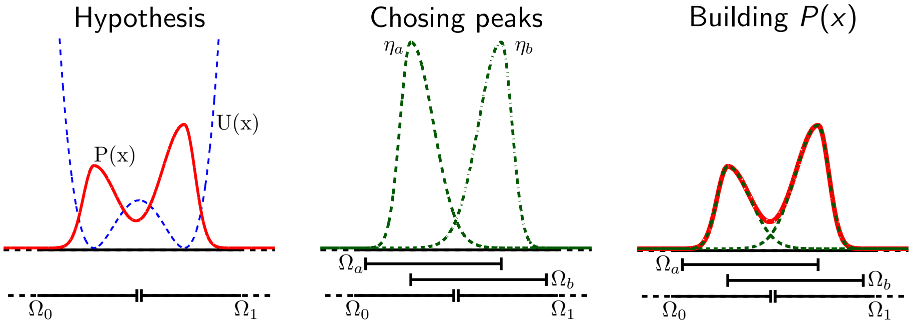
- peak functions.
- thay are non-negative with supports Ω_a, Ω_b respectively.
- $\int_{\Omega_a} \eta_a dx = \int_{\Omega_b} \eta_b dx = 1$

Thanks to this last one

- $P_a + P_b = 1$
- P_a and P_b are the probability that a microstate x belongs to η_a or η_b



The sample structure of $P(x)$ - [2]



Note that:

- Each peak has its own shape;
- Peaks significatively overlap near the boundary between Ω_0, Ω_1;
- Ω_a and Ω_b may not coincide with Ω_0 and $\Omega_1 \Rightarrow (P_a, P_b)$ is not (P_0, P_1).

$$P_0 = P_a \int_{\Omega_a \cap \Omega_0} \eta_a dx + P_b \int_{\Omega_b \cap \Omega_0} \eta_b dx$$

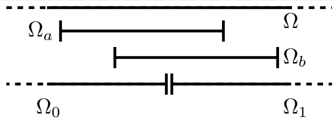
$$P_1 = P_a \int_{\Omega_a \cap \Omega_1} \eta_a dx + P_b \int_{\Omega_b \cap \Omega_1} \eta_b dx$$

A simple formula for S_{cond}

$$S_{cond} = S_G - S_S = S_{ex} + S_{sh} + S_{ov}$$

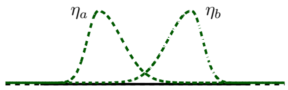
$$S_{ex} = -K_B P_a \log P_a - K_B P_b \log P_b + K_B P_0 \log P_0 + K_B P_1 \log P_1$$

S_{ex} is the entropic measure of the error committed by exchanging (P_a, P_b) with (P_0, P_1).



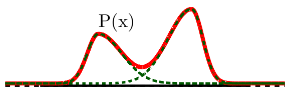
$$S_{sh} = P_a S_a + P_b S_b$$
$$S_a = -K_B \int_{\Omega_a} \eta_a \log \eta_a dx$$

S_{sh} gives the entropic measure of η_a and η_b shapes



$$S_{ov} = P_a I\left(\eta_a, \eta_b, \frac{P_b}{P_a}\right) + P_b I\left(\eta_b, \eta_a, \frac{P_a}{P_b}\right)$$
$$\frac{I(\eta_a, \eta_b, q)}{K_B} = -\int_{\Omega_a \cap \Omega_b} \eta_a \log\left(1 + q\frac{\eta_b}{\eta_a}\right) dx$$

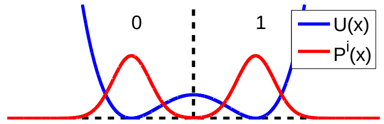
S_{ov} is the entropic measure of η_a and η_b overlap in $P(x)$



Gaussian Example

We reset a bit of information

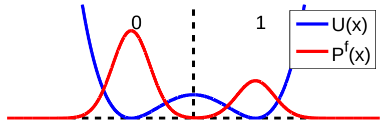
Initial state



$$P^i(x) = \frac{1}{2}\frac{e^{\frac{(x+h)^2}{2}}}{\sqrt{2\pi}} + \frac{1}{2}\frac{e^{\frac{(x-h)^2}{2}}}{\sqrt{2\pi}}$$

$$P^i_0 = P^i_1 = \frac{1}{2}$$

Final state

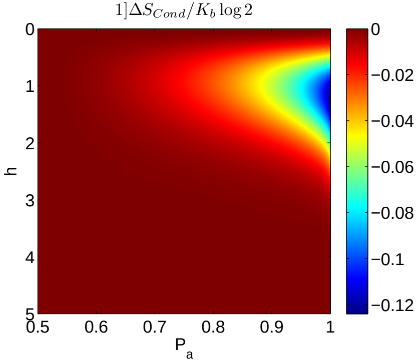


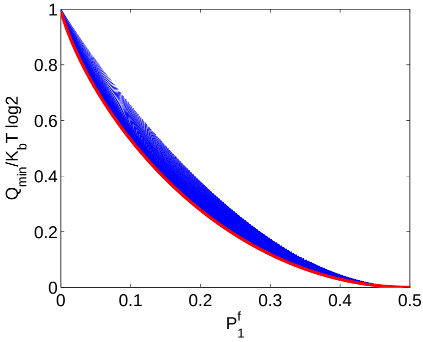
$$P^f(x) = P_a\frac{e^{\frac{(x+h)^2}{2}}}{\sqrt{2\pi}} + (1-P_a)\frac{e^{\frac{(x-h)^2}{2}}}{\sqrt{2\pi}}$$

$$P^f_0 = \frac{1+(2P_a-1)\,\mathrm{erf}(\frac{h}{\sqrt{\pi}})}{2}$$

Free parameters: P_a and h ($h \approx \sqrt{\frac{\Delta U}{K_B T}}$)

Some results





- $\Delta S_{cond} \neq 0$
- for $h \approx 1$ $\left[\frac{\Delta U}{K_B T} \approx 1\right]$, ΔS_{cond} can become up to 25% of ΔS_G.
- ΔS_S is insufficient to characterize minimum heat production

Conclusions

- Brief introduction to ΔS_{cond}, writing it in a simple and intuitive way for bistable systems.
- Discussed the implications of ΔS_{cond} to minimum heat production with a simple example based on gaussian peaks.

Further readings

- D. Chiuchiú, M. C. Diamantini, L. Gammaitoni. *Role of conditional entropy in experimental tests of Landauer Principle.*, arXiv:1406.2562
- T. Sagawa *Thermodynamic and Logical Reversibilities Revisited* J. Stat. Mech. (2014) P03025